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On the forced oscillations of a small gas bubble in a spherical liquid-filled flask

By R. I. NIGMATULIN[1], I. Sh. AKHATOV[1], N. K. VAKHITOVA[1] and R. T. LAHEY, Jr[2]

[1] Tyumen Institute of Mechanics of Multiphase Systems (TIMMS), Ufa-Bashkortostan Branch of the Russian Academy of Sciences, 450000 Ufa, K. Marx 6, Russia
[2] Center for Multiphase Research, Rensselaer Polytechnic Institute, Troy, NY 12180-3590, USA

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A spherically-symmetric problem is considered in which a small gas bubble at the centre of a spherical flask filled with a compressible liquid is excited by small radial displacements of the flask wall. The bubble may be compressed, expanded and made to undergo periodic radial oscillations. Two asymptotic solutions have been found for the low-Mach-number stage. The first one is an asymptotic solution for the field far from the bubble, and it corresponds to the linear wave equation. The second one is an asymptotic solution for the field near the bubble, which corresponds to the Rayleigh–Plesset equation for an incompressible fluid. For the analytical solution of the low-Mach-number regime, matching of these asymptotic solutions is done, yielding a generalization of the Rayleigh–Plesset equation. This generalization takes into account liquid compressibility and includes ordinary differential equations (one of which is similar to the well-known Herring equation) and a difference equation with both lagging and leading time. These asymptotic solutions are used as boundary conditions for bubble implosion using numerical codes which are based on partial differential conservation equations. Both inverse and direct problems are considered in this study. The inverse problem is when the bubble radial motion is given and the evolution of the flask wall pressure and velocity is to be calculated. The inverse solution is important if one is to achieve superhigh gas temperatures using non-periodic forcing (Nigmatulin *et al.* 1996). In contrast, the direct problem is when the evolution of the flask wall pressure or velocity is given, and one wants to calculate the evolution of the bubble radius. Linear and nonlinear periodic bubble oscillations are analysed analytically. Nonlinear resonant and near-resonant periodic solutions for the bubble non-harmonic oscillations, which are excited by harmonic pressure oscillations on the flask wall, are obtained. The applicability of this approach bubble oscillations in experiments on single-bubble sonoluminescence is discussed.

1. Introduction

Let us consider the spherically-symmetric radial flow of a compressible liquid in a spherical flask which has a small spherical gas bubble located at the centre (figure 1). The instantaneous radius of the flask is $R(t)$ and that of the bubble is $a(t)$. Let us assume that the bubble is very small:

$$a \ll R. \tag{1.1}$$

We further assume that the flask wall is undergoing small-amplitude high-frequency

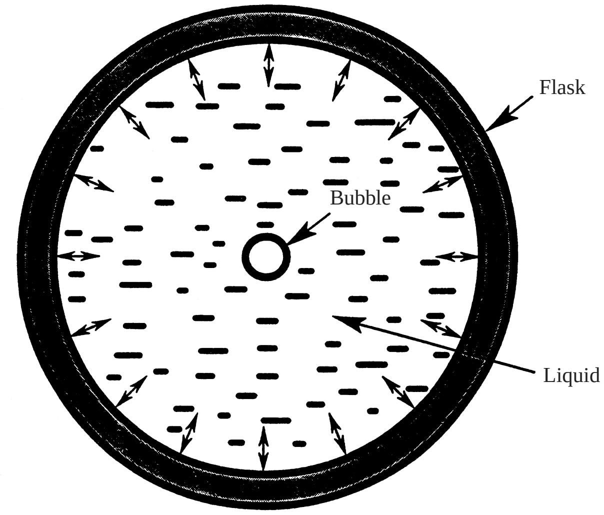

FIGURE 1. Schematic of the spherically-symmetric problem of an oscillating gas bubble in a liquid-filled flask.

spherically-symmetric displacements, δ, where

$$\delta(t,R) \ll R(t). \tag{1.2}$$

If the pressure disturbances are not large enough to appreciably change the density of the liquid and the wavelength of the pressure wave in the liquid, λ, is much larger than the bubble's radius a, then for the mathematical modelling of the process one may use the approximation of an incompressible liquid, for which the momentum equation becomes the well-known Rayleigh–Plesset equation (Rayleigh 1917; Knapp, Dayly & Hammitt 1970; Nigmatulin 1991):

$$a\frac{\mathrm{d}w_a}{\mathrm{d}t} + \tfrac{3}{2}w_a^2 = \frac{p_a - p_\infty}{\rho}, \quad w_a = \frac{\mathrm{d}a}{\mathrm{d}t}, \quad p_a = p_g(a) - \frac{2\sigma}{a} - \frac{4\mu w_a}{a}, \tag{1.3a–c}$$

where ρ, μ, σ are the density, viscosity and surface tension of the liquid, respectively, w_a and p_a are the radial velocity and the pressure of the liquid on the bubble's interface, p_g is the pressure of the gas in the bubble, and p_∞ is the pressure of the liquid far from the bubble.

Previously the influence of liquid compressibility on the volume oscillations of a gas bubble was considered taking into account acoustic radiation damping (Herring 1941; see also Knapp *et al.* 1970; Nigmatulin 1991), which leads to the well-known approximate equation

$$\left(1-\alpha_1\frac{w_a}{C}\right)a\frac{\mathrm{d}w_a}{\mathrm{d}t} + \frac{3}{2}\left(1-\alpha_2\frac{w_a}{C}\right)w_a^2 = \left(1+\alpha_3\frac{w_a}{C}\right)\frac{p_a - p_\infty}{\rho} + \frac{a}{\rho C}\frac{\mathrm{d}(p_a - p_\infty)}{\mathrm{d}t}. \tag{1.4}$$

This equation looks like the Rayleigh–Plesset equation but it has additional terms: $(a/\rho C)\mathrm{d}(p_a - p_\infty)/\mathrm{d}t$ and w_a/C, where C is the sound speed in the liquid, and $\alpha_i (i = 1, 2, 3)$ are the coefficients ($\alpha_i \sim 1$).

A systematic theory has been presented by Prosperetti & Lezzi (1986) and Lezzi & Prosperetti (1987) for the radial motion of a spherical bubble in an infinite weakly compressible liquid. A whole family of equations for bubble oscillations has been obtained including (1.4) and other authors' equations as specific cases. All these equations are shown to be equivalent as they have the same order of accuracy for Mach number, $M = w_a/C$. In deriving these equations, the assumption was made that bubble oscillations did not affect the outer acoustic field pressure at infinity. Indeed, the fact that the fluid is unbounded permits one to consider the problem of bubble oscillations separately from the acoustic problem.

The present paper deals with the problem of the oscillations of a gas bubble in a liquid-filled flask of finite size, in which the flask wall is used to excite the liquid. As shown, the problem may be reduced to an equation for the evolution of bubble radius which is similar to the Herring equation. However, instead of the pressure far from the bubble, p_∞, another pressure, p_I, appears, which is the incident pressure. Significantly, the incident pressure differs both from the pressure at local infinity, p_∞, and from the pressure on the flask wall, p_R. In fact, it takes into account interacting waves and must be calculated using a difference differential equation including the bubble radius, $a(t)$, and the pressure on the flask, $p_R(t)$.

2. Problem formulation

The posing of the problem for the spherically-symmetric radial flow (i.e. a radial velocity field $w(t,r)$) of a compressible liquid around a spherical bubble includes the differential equations of mass and momentum, the barotropic equation of state for pressure p (which only depends on liquid density ρ), the boundary conditions on the bubble's interface ($r = a$) and on the wall of the flask ($r = R$), and the initial conditions at $t = 0$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho w)}{\partial r} + \frac{2\rho w}{r} = 0, \tag{2.1}$$

$$\rho\frac{\partial w}{\partial t} + \rho w\frac{\partial w}{\partial r} + \frac{\partial p}{\partial r} = 0, \tag{2.2}$$

$$p = p(\rho); \tag{2.3}$$

$$r = R: \quad p = p_R(t) \quad \text{or} \quad w = w_R(t); \tag{2.4}$$

$$r = a: \quad p = p_a, \quad w = w_a \quad \left(p_a = p_g(a) - \frac{2\sigma}{a} - 2\mu\frac{\partial w}{\partial r}, \quad w_a = \frac{\mathrm{d}a}{\mathrm{d}t} \equiv a'\right); \tag{2.5}$$

$$t = 0: \quad a = a_0, \quad a' = w_0. \tag{2.6}$$

The boundary condition at $r = R$ corresponds to interaction with some body (e.g. a piezoelectric transducer) contacting with the wall of the flask and $p_R(t)$ or $w_R(t)$ must be given.

Below it is shown that the space between the bubble's interface and the internal surface of the flask consists of three zones:

(1) The far field, or external region, where weak compressibility of the liquid is essential but convective displacements of the liquid, $\delta(t,r)$, are small ($\delta \ll r$), and the nonlinear convective terms in the mass and momentum conservation equations are negligibly small. In this zone, the motion of the liquid has an acoustic wave character with finite speed of disturbance propagation determined by a constant sound speed at the initial state.

(2) The near field of the bubble, or internal region, where the liquid may be considered incompressible and motion occurs only because of compression and expansion of the bubble, and nonlinear inertia forces, because of the convective accelerations, are essential.

(3) An intermediate zone where both the liquid compressibility and nonlinear inertia forces, because of the convective accelerations, are essential.

In the first two zones it is possible to obtain asymptotic analytical solutions.

3. External asymptotic solution far from the bubble

In the external zone far from the bubble ($r^2 \gg a^2$) convective derivatives of density and velocity are small.

$$w\frac{\partial\rho}{\partial r} \ll \frac{\partial\rho}{\partial t}, \quad w\frac{\partial w}{\partial r} \ll \frac{\partial w}{\partial t}, \tag{3.1}$$

which follows from the estimations

$$w\frac{\partial\rho}{\partial r} \sim \frac{w\Delta\rho}{\lambda_R}, \quad \frac{\partial\rho}{\partial t} \sim \frac{\Delta\rho}{t_R}, \quad w\frac{\partial w}{\partial r} \sim \frac{w^2}{\lambda_R}, \quad \frac{\partial w}{\partial t} \sim \frac{w}{t_R} \tag{3.2}$$

$$\left(w \sim \frac{w_R R^2}{r^2}, \quad \lambda_R = Ct_R\right).$$

Here λ_R and t_R are the length and the period of the wave disturbance in the liquid in the external zone, and w_R is a characteristic velocity of the liquid on the flask wall. We note that t_R is the characteristic period of the oscillations of the flask. The ratio of the nonlinear convective and time derivatives is proportional to the ratio of the flask displacement ($w_R t_R$) to the length of the flask wave (λ_R), or the characteristic Mach number, M_R, of flask motion:

$$w\frac{\partial\rho/\partial r}{\partial\rho/\partial t} \sim w\frac{\partial w/\partial r}{\partial w/\partial t} \sim \frac{wt_R}{\lambda_R} \sim \frac{R^2 w_R t_R}{r^2\lambda_R} = \frac{R^2}{r^2}\frac{w_R}{C} = \frac{R^2}{r^2}M_R \ll 1, \tag{3.3}$$

which is supposed to be small for $r \sim R$ (i.e. we study the low-Mach-number regime for small displacements of the flask). Thus one may neglect the nonlinear convective terms in the momentum and mass conservation equations, (2.1)–(2.3), compared with the temporal acceleration terms, and these equations for velocity and pressure in the 'external' field (i.e. $r \gg a,\ w \approx w_{ex},\ p \approx p_{ex}$) reduce to the following linear acoustic equations:

$$\frac{\partial^2\varphi_{ex}}{\partial t^2} = C^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_{ex}}{\partial r}\right), \quad C^2 = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0, \quad p_{ex} = p_0 - \rho\frac{\partial\varphi_{ex}}{\partial t}, \quad w_{ex} = \frac{\partial\varphi_{ex}}{\partial r}. \tag{3.4}$$

Here density, ρ, and sound speed, C, correspond to an initial state of rest.

The solution to the equation (3.4) is well-known and thus the external asymptotic solution for the velocity potential can be expressed as

$$\varphi_{ex} = \frac{1}{r}\left[\psi_1\left(t - \frac{r}{C}\right) + \psi_2\left(t + \frac{r}{C}\right)\right], \tag{3.5}$$

where ψ_2 and ψ_1 characterize the incident and the reflected acoustic waves, respectively.

4. Internal asymptotic solution near the bubble

The internal zone near the bubble has a radius, $r = O(a)$, which is small compared to the radius of the flask R. In this zone, during the characteristic time of bubble compression, t_a, the displacements of the liquid are about one bubble radius, $w_a t_a \sim a$. For low Mach numbers near the bubble, $M_a \equiv w_a/C \ll 1$, this assumption corresponds to long waves near the bubble: $\lambda_a \sim Ct_a \gg a$. In such a case, if the relative change of the liquid density near the bubble is assumed to be small, then in the internal zone ($w = w_{in}$, $\varphi = \varphi_{in}$) the asymptotic solution of an incompressible fluid is valid (see also Landau & Lifshitz 1959; Feynman, Leighton & Sands 1964), thus equation (2.1) yields

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi_{in}}{\partial r}\right) = 0, \quad w_{in} = \frac{\partial \varphi_{in}}{\partial r}. \tag{4.1}$$

The solution of this equation together with the boundary conditions on the bubble interface ($r = a$: $w = w_a \equiv a'$) is

$$\varphi_{in} = -\frac{a'a^2}{r}, \quad w_{in} = \frac{a'a^2}{r^2}. \tag{4.2}$$

The Bernoulli integral in this case includes a component corresponding to a nonlinear convective inertia force ($w^2/2$), and has the form

$$\frac{\partial \varphi_{in}}{\partial t} + \frac{w_{in}^2}{2} + \frac{p_{in}}{\rho} = F_{in}(t). \tag{4.3}$$

This equation in conjunction with equation (4.2) and the boundary condition (2.5) on the bubble interface ($r = a$) leads to the Rayleigh–Plesset equation (1.3), where p_∞ is the pressure at 'intermediate infinity' ($r = r_\infty$), which is far from the bubble, and far from the surface of the flask, i.e. $a \ll r_\infty \ll R$. Using this integral, one can find the asymptotic solution for the pressure distribution in the internal zone:

$$p_{in} = p_a - \rho(aa'' + \tfrac{3}{2}(a')^2) + \frac{\rho}{r}(a^2a')' - \frac{\rho}{2}\frac{a^4(a')^2}{r^4}. \tag{4.4}$$

This result will be useful in the next section for the matching procedure.

5. Matching of the asymptotic solutions

To obtain an equation for the radial motion of gas bubbles for a given excitation on the flask (2.4), and taking into account liquid compressibility, it is necessary that the asymptotic solutions in the external and internal zones be matched in the intermediate zone. For the internal asymptotic solution the intermediate zone is at infinity ($r \to \infty$), but for the external asymptotic solutions the intermediate zone is near the centre of the flask ($r \to 0$). As a matching condition in the intermediate zone the equality of volumetric flow $4\pi r^2 w$ and pressure p was imposed for the internal and external asymptotic solutions. Thus the radial coordinate in the asymptotic solution for the internal zone ($r \sim a$) should tend to infinity ($r \to \infty$) and in the asymptotic solution for the external zone ($r \sim R$) it should tend to zero ($r \to 0$):

$$4\pi r^2 w_{in}\big|_{r\to\infty} = 4\pi r^2 w_{ex}\big|_{r\to 0}, \quad p_{in}|_{r\to\infty} = p_{ex}|_{r\to 0}. \tag{5.1}$$

Taking into account that the volumetric flow for an incompressible fluid depends only on time:

$$4\pi r^2 w_{in} = 4\pi r^2 \frac{\partial \varphi_{in}}{\partial r} = 4\pi a^2 \frac{\mathrm{d}a}{\mathrm{d}t} \equiv 4\pi Q(t) \quad (Q \equiv a^2 a'), \tag{5.2}$$

we may rewrite the first equation in (5.1) as

$$r^2 \frac{\partial \varphi_{ex}}{\partial r}\bigg|_{r\to 0} = a^2 \frac{\mathrm{d}a}{\mathrm{d}t}. \tag{5.3}$$

Thus,

$$r^2 \frac{\partial \varphi_{ex}}{\partial r}\bigg|_{r\to 0} = -\left[\psi_1\left(t-\frac{r}{C}\right)+\psi_2\left(t+\frac{r}{C}\right)\right] + \frac{r}{C}\left[-\psi_1'\left(t-\frac{r}{C}\right)+\psi_2'\left(t+\frac{r}{C}\right)\right]\bigg|_{r\to 0}$$
$$\to -\psi_1(t)-\psi_2(t), \tag{5.4}$$

where the prime corresponds to the derivative. Then from (5.3), (5.4) the relation between the incident and reflected waves is

$$\psi_1 = -\psi_2 - a^2 a'. \tag{5.5}$$

Thus from (3.5), (4.2) one has the asymptotic solution for the velocity potential in the external and internal zones as

$$\varphi_{ex} = \frac{1}{r}\left[\psi_2\left(t+\frac{r}{C}\right)-\psi_2\left(t-\frac{r}{C}\right)-Q\left(t-\frac{r}{C}\right)\right], \quad \varphi_{in} = -\frac{Q(t)}{r}. \tag{5.6}$$

Using (3.4), (5.6), we find the asymptotic solutions for pressure distributions in the external zone:

$$p_{ex} = p_0 - \rho\frac{\partial \varphi_{ex}}{\partial t} = p_0 - \frac{\rho}{r}\left[\psi_2'\left(t+\frac{r}{C}\right)-\psi_2'\left(t-\frac{r}{C}\right)-Q'\left(t-\frac{r}{C}\right)\right]. \tag{5.7}$$

Substitution of (4.4) and (5.7) into the second matching condition in (5.1) leads to the long wave approximation of the equation for bubble oscillations in a compressible liquid:

$$aa'' + \tfrac{3}{2}a'^2 = \frac{p_a - p_0}{\rho} + \frac{1}{C}\left[2\psi_2''(t) + Q''(t)\right]. \tag{5.8}$$

Here the term $Q''(t)/C$ is responsible for the influence of bubble oscillations on the reflected pressure wave.

The pressure on the flask wall ($r = R$) may be expressed using the external asymptotic solution (5.7):

$$p_R(t) = p_0 - \frac{\rho}{R}\left[\psi_2'\left(t+\frac{R}{C}\right)-\psi_2'\left(t-\frac{R}{C}\right)-Q'\left(t-\frac{R}{C}\right)\right]. \tag{5.9}$$

From equations (1.3) and (5.8) the expression for the pressure, p_∞, at the internal 'infinity' (i.e. far from the bubble, but near the centre compared with the radius of the flask) is

$$p_\infty = p_0 - \frac{\rho}{C}\left[2\psi_2''(t) + Q''(t)\right]. \tag{5.10}$$

Equations (5.8) and (5.9) are the expressions sought in the matching procedure.

6. The system of equations for bubble motion

It is important that (5.8) depends on Q'', having the third derivative of the bubble radius, $a'''(t)$. The dependence of the solution for bubble radius, $a(t)$, on the third derivative, $a'''(t)$, was noted previously by Prosperetti & Lezzi (1986), and Prosperetti (1994). However, for the low-Mach-number, or long-wave, approximation one may use the simple asymptotic solution without the third derivative. Indeed, the components of (5.8) may be estimated as

$$aa'' \sim (a')^2 \sim \frac{a^2}{t_a^2}, \quad \frac{1}{C}Q''(t) \equiv \frac{1}{C}(a^2a')'' \sim \frac{a^3}{t_a^3C}, \tag{6.1}$$

and their ratio is determined by a small parameter for the case of the assumed long-wave, or low-Mach-number regime in the internal zone (see the discussion in the § 4):

$$\frac{Q''/C}{aa''} \sim \frac{Q''/C}{Q'/a} \sim \frac{a}{t_aC} \sim \frac{a}{\lambda_a} \sim \frac{a'}{C} \sim \varepsilon_a \ll 1. \tag{6.2}$$

Taking into account

$$\frac{Q'}{a} \equiv aa'' + 2a'^2, \quad aa'' + \tfrac{3}{2}a'^2 \equiv \tfrac{Q'}{a} - \tfrac{1}{2}a'^2, \tag{6.3}$$

we may rewrite (5.8), introducing the incident (wave) pressure p_I, as

$$\frac{Q'}{a} - \frac{a'^2}{2} = \frac{p_a - p_1}{\rho} + \frac{Q''}{C}, \quad p_I(t) = p_0 - \frac{2\rho}{C}\psi_2''. \tag{6.4}$$

Note that the so-called incident pressure p_I is the regular (i.e. non-singular) part of the liquid pressure (5.9), extrapolated to the centre of the bubble.

Suppose that the term $Q''(t)/C$ is small in (6.2). This expression may then be simplified:

$$Q' = a\left[\frac{a'^2}{2} + \frac{p_a - p_I}{\rho}\right](1 + O(\varepsilon_a)). \tag{6.5}$$

Differentiating with respect to time, t, one obtains the long-wave and low-Mach-number approximation for $Q''(t)/C$:

$$\frac{Q''}{C}(1 + O(\varepsilon_a)) = \left[aa'' + \frac{a'^2}{2} + \frac{p_a - p_I}{\rho_0}\right]\frac{a'}{C} + \frac{a}{\rho C}\frac{\mathrm{d}}{\mathrm{d}t}[p_a - p_I]. \tag{6.6}$$

Substituting (6.6) into (5.8) and neglecting the values of $O(\varepsilon_a^2)$ we obtain

$$\left(1 - \frac{a'}{C}\right)aa'' + \frac{3}{2}\left(1 - \frac{a'}{3C}\right)a'^2 = \left(1 + \frac{a'}{C}\right)\frac{p_a - p_I}{\rho} + \frac{a}{\rho C}\frac{\mathrm{d}}{\mathrm{d}t}[p_a - p_I]. \tag{6.7}$$

For small Mach number ($M_a \equiv a'/C$) the correction terms in the square brackets of this equation may be neglected.

Finally, the evolution of the radius of the spherical bubble in the centre of a spherical flask filled with a viscous, weakly compressible, liquid excited by radial displacements of the flask in the long-wave regime, for low Mach number, is described by the following system of equations resulting from (5.9), (6.4) and (6.7):

$$aa'' + \tfrac{3}{2}a'^2 = \frac{p_a - p_I}{\rho} + \frac{a}{\rho C}\frac{\mathrm{d}}{\mathrm{d}t}[p_a - p_I], \tag{6.8}$$

$$p_R(t) = p_0 - \frac{\rho}{R}\left[\psi_2'\left(t + \frac{R}{C}\right) - \psi_2'\left(t - \frac{R}{C}\right) - Q'\left(t - \frac{R}{C}\right)\right], \tag{6.9}$$

$$p_I(t) \equiv p_0 - \frac{2\rho}{C}\psi_2'', \quad Q(t) \equiv a^2a'. \tag{6.10}$$

This system of coupled ordinary difference-differential equations (6.8) and (6.9) has both lagging (retarding) and leading potentials. It is closed by specifying the density

ρ, and sound speed C of the liquid. The value of the liquid pressure on the bubble's interface, p_a, is calculated from the boundary condition on the bubble's interface ($r = a$), where it is necessary to use the surface tension σ and viscosity μ of the liquid (see (1.3c)) and an equation of state for the gas bubble $p_g(a)$ (in particular, the polytropic law with an exponent κ):

$$p_g = (p_0 + 2/a_0)\left(a/a_0\right)^{-3\kappa} \quad (1 < \kappa < \gamma), \tag{6.11}$$

where γ is the adiabatic exponent of the gas.

One of the boundary conditions (2.4) on the flask wall ($r = R$) determines the external excitation of the system. The initial conditions are determined by (2.6).

Notice that (6.8) is similar in form to the Herring equation (1.4). However in (6.8) instead of the liquid pressure at infinity, p_∞, the incident pressure, p_I, appears.

It is important to note that the incident pressure, p_I, is not the same as p_∞, nor the flask wall pressure $p_R(t)$. The incident pressure p_I, given by (6.10), takes into account the convergent waves (given by the function $\psi_2(t)$) coming to the bubble from the flask.

One might suppose that the incident pressure p_I corresponds to the pressure of the liquid at the centre of the flask when there is no bubble. However, this is true only for the trivial case when the incident pressure is constant (p_I = const), i.e. for free oscillations. For forced oscillations the incident pressure p_I is influenced and compensated by bubble expansion or compression (note the Q' term in (6.9)). In particular, for the periodic flask resonance regime the liquid pressure at the centre of the flask without a bubble may be infinite; however p_I is always finite (see § 9). It is important to note that the incident pressure p_I is calculated only after solving the coupled problem, i.e. equations (6.8)–(6.10), for interacting flask and bubble oscillations.

7. Inverse problem

Interestingly, using equations (6.8)–(6.10), it is easy to solve the inverse problem; that is knowing the evolution of the bubble radius $a(t)$ to calculate the required evolution of the pressure on the flask wall $p_R(t)$. The inverse solution is important if one is to achieve superhigh gas temperatures using non-periodic forcing (Nigmatulin *et al.* 1996). In order to obtain the inverse solution it is necessary to first calculate the incident pressure $p_I(t)$ using the linear (relatively to p_I) differential equation (6.8):

$$\frac{\mathrm{d}p_I}{\mathrm{d}t} - \frac{C}{a}p_I = \frac{\mathrm{d}}{\mathrm{d}t}p_a + \frac{C}{a}p_a - \rho\frac{C}{a}(aa'' + \tfrac{3}{2}a'^2). \tag{7.1}$$

Then for the forcing function ψ_2, there is an integral, following from (6.12):

$$\psi_2'(t) = \frac{C}{2\rho}\int_0^t [p_0 - p_I(t')]\,\mathrm{d}t', \tag{7.2}$$

and finally, knowing $\psi_2'(t)$ and $Q'(t) \equiv a^2a'' + 2(a')^2$, one may calculate the pressure on the flask wall $p_R(t)$ using the difference equation (6.9).

The 'semi-inverse' problem is when, knowing the incident pressure $p_I(t)$, the evolution of the bubble radius $a(t)$ and $Q'(t)$ are calculated solving the differential equation (6.10). Then the evolution of the pressure distribution, $p_{ex}(t, r)$, in the external field, in particular the evolution of the flask pressure, $p_R(t)$ at $r = R$, is calculated using (7.2) and (5.7):

$$p_{ex}(t, r) = p_0 + \frac{C}{2r}\int_{t-r/c}^{t+r/c} [p_I(t') - p_0]\,\mathrm{d}t' + \frac{\rho}{r}Q'(t - r/c). \tag{7.3}$$

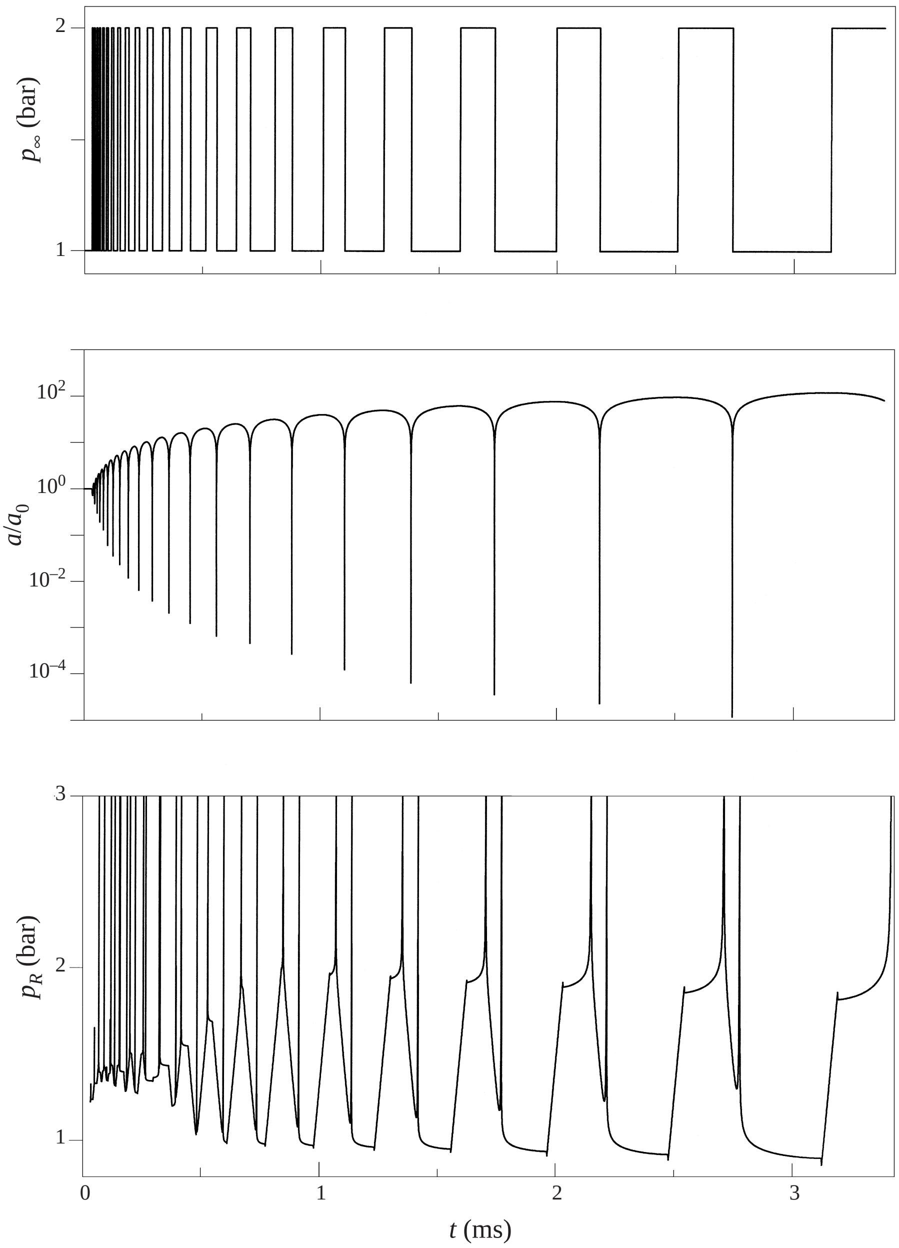

FIGURE 2. 'Basketball dribbling' regime exciting of a gas bubble ($a_0 = 35\,\mu$m) in water ($p_0 = 1$ bar, $T_0 = 300\,$K) in a spherical flask ($R = 5\,$cm) for relatively small forcing compression pressure ($p_\infty^{max} = 2$ bar).

Similarly, the evolution of the pressure in the internal field is calculated using (4.4):

$$p_{in}(t,r) = p_I(t) + \frac{a}{C}\frac{\mathrm{d}p_I(t)}{\mathrm{d}t} - \frac{a}{C}\frac{\mathrm{d}p_a}{\mathrm{d}t} + \frac{\rho}{r}(a^2a')' - \frac{\rho}{2}\frac{a^4(a')^2}{r^4}. \tag{7.4}$$

The inverse problem solution was used for calculating the flask's wall pressure evolution, $p_R(t)$, required to give a coordinated non-periodic resonant 'basketball dribbling regime' (Nigmatulin *et al.* 1995, 1996; Nigmatulin & Lahey 1995). This method of excitation allows the system kinetic energy to be progressively increased by moderate-amplitude forcing of the flask pressure Δp_R (see figure 2) due to coordination between the bubble radius evolution $a(t)$, evolution of pressure at local infinity, $p_\infty(t)$, and pressure at the flask wall, $p_R(t)$.

8. The direct problem

The direct problem is as follows: knowing the evolution of the flask wall pressure $p_R(t)$ or flask wall velocity $w_R(t)$, to calculate the evolution of the bubble radius $a(t)$. In the general case the solution of the direct problem may be obtained by the numerical integration of the system of partial differential equations (2.1)–(2.5). The direct numerical simulation of the process of forming the periodic regime for very small bubbles ($a \ll R$) often requires an enormous quantity of computer time.

For the case of long-wave disturbances and weak compressibility of the liquid, we may use a much more 'economical' analytical approach taking into account the internal asymptotic solution near the bubble and external asymptotic solution far from the bubble.

An analytical approach to the direct problem follows after differentiating with respect to time the equation for the flask wall pressure (6.9):

$$\frac{\mathrm{d}p_R}{\mathrm{d}t} = -\frac{\rho}{R}\left[\psi_2''\left(t+\frac{R}{C}\right) - \psi_2''\left(t-\frac{R}{C}\right) - Q''\left(t-\frac{R}{C}\right)\right], \tag{8.1}$$

Taking into account that (see (6.10))

$$-\psi_2''(t) = \frac{C}{2\rho}[p_I(t) - p_0], \tag{8.2}$$

one obtains a recursion formula for calculating the incident pressure p_I:

$$p_I\left(t+\frac{R}{C}\right) = p_I\left(t-\frac{R}{C}\right) + \frac{2R}{C}p_R'(t) - \frac{2\rho}{C}Q''\left(t-\frac{R}{C}\right). \tag{8.3}$$

This equation allows one to calculate the values of the incident pressure p_I at a later time $(t+R/C)$ by using the same pressure p_I, the derivative of the flask pressure $p_R'(t)$, and the second derivative of Q, at earlier times (i.e. t and $t-R/C$). As a result if one knows the evolution of the bubble radius during the time period $\Delta t = R/C$ (in particular it may be the state of rest before excitation began) then the last recurrent equation determines the incident pressure p_I for any later time until the flask pressure is given.

Using simplified expression (6.6) for Q'', for the long-wave and low-Mach-number approximation, one may write the recurrent equation for the incident pressure as

$$p_I\left(t+\frac{R}{C}\right) = p_A + p_B, \tag{8.4a}$$

$$p_A = p_I\left(t-\frac{R}{C}\right) + \frac{2R}{C}p_R'(t), \tag{8.4b}$$

$$p_B = -2\left\{\left[\rho\left(aa''+\frac{a'^2}{2}\right) + (p_a(a,a') - p_I)\right]\frac{a'}{C} + \frac{a}{C}\frac{\mathrm{d}}{\mathrm{d}t}[p_a(a,a') - p_I]\right\}\bigg|_{t-R/C}. \tag{8.4c}$$

Thus the incident pressure consists of two components. The first, p_A, is the pressure corresponding to an acoustical process in the flask filled with a liquid but without

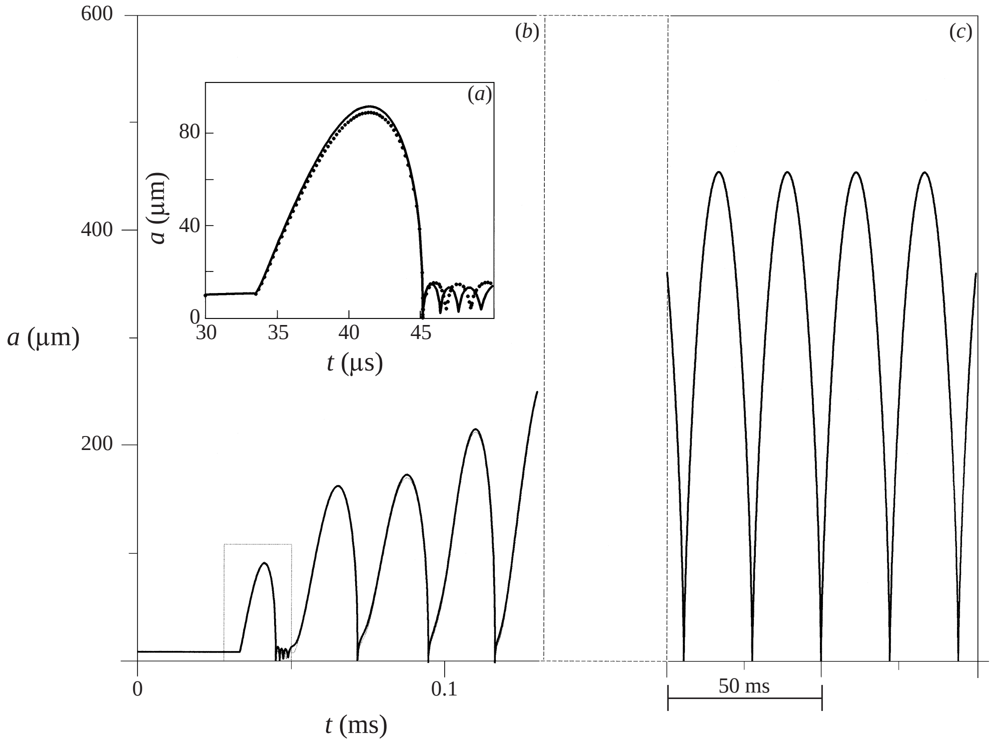

FIGURE 3. Excitation of adiabatic ($\kappa = \gamma$) gas bubble ($a_0 = 10\,\mu$m) oscillations in a flask ($R = 5\,$cm) filled with water ($p_0 = 1$ bar, $T_0 = 300\,$K) when the flask wall produces sinusoidal pressure oscillations of amplitude $\Delta p_R^o = 0.25$ bar, and frequency, $f = f_3 = 45\,$kHz. (*a*) Comparison with the numerical solution of Moss *et al.* (1994) given by the dotted line. (*b*) The first few periods of bubble oscillations. (*c*) The periodic regime which corresponds to the resonant case ($k = 3$ or $f = f_3$).

a bubble. The second, p_B, is the pressure due to reaction on the flask's wall due to bubble oscillations.

In Moss *et al.* (1994) a numerical code for partial differential conservation equations was used to calculate the bubble's behaviour not only for the bubble implosion stage but for the long-wavelength and low-Mach-number stage as well. In this numerical analysis the asymptotic solution of the incompressible liquid near the bubble was not used. The direct problem was solved for a sinusoidal pressure change on the flask:

$$p_R(t) = \begin{cases} p_0 & \text{for} \quad t \leqslant 0 \\ p_0 - \Delta p_R^o \sin \omega t & \text{for} \quad t > 0, \end{cases} \tag{8.5}$$

where Δp_R^o is the amplitude of pressure on the flask wall, $\omega = 2\pi f$ is angular frequency. The bubble radius $a(t)$ was calculated, but only for the first oscillation and not for the periodic regime, which takes place only after many oscillations have washed out the initial condition. Significantly, the periodic regime is what is normally measured in experiments.

In figure 3 the results of our calculations are presented (solid line). The first oscillation agrees with the Moss *et al.* (1994) calculation (dotted line). One can see that the second oscillation differs essentially from the first, the third differs from the second and so on. For the isothermal behaviour of the bubble, radius oscillations practically coincide with those for the adiabatic behaviour, with the exception of the moments of maximum compression: for the isothermal behaviour the depth of the collapse is essentially less. And only after many oscillations ($t \gg \omega^{-1}$) does a periodic regime take place. This is connected with the inertia of the liquid in the flask, $\sim \rho R^3$. In fact, it is the mass of the liquid that delays beginning the periodic regime of pressure at local infinity, p_∞. The delay of the periodic regime of bubble oscillations after the beginning the periodic regime for p_∞ is much smaller because it determined by the virtual mass of the liquid around the bubble, $\sim \rho a^3 \ll \rho R^3$.

It follows that the most interesting solution should be for the periodic regime. However, a direct numerical solution of the initial value problem for the periodic regime is very time consuming.

It should be noted also that heat transfer phenomena in the bubble are of great importance. The detailed analysis of the temperature effects in the bubble will be presented in a separate paper (Nigmatulin, Akhatov & Vakhitova 1999*a*; Nigmatulin *et al.* 1999*b*). Here we provide some estimations to decide if it is possible to use an isothermal or adiabatic approximation as the equation of state for the gas bubble.

The gas temperature on the bubble's interface is practically constant and equal to the temperature of the liquid, T_0. There is a thermal boundary layer near the interface where the gas temperature varies from T_0 on the interface to the bulk temperature of the gas T, which is varying with time because of the compression or expansion of the gas bubble.

A well-known estimation of the transient thermal boundary layer thickness in the bubble is given by

$$\delta r^{(T)} \approx \pi \sqrt{\nu_g^{(T)} \delta t}, \quad \nu_g^{(T)} = \lambda_g / (\rho_g c_p). \tag{8.6}$$

Here $\nu_g^{(T)}$, λ_g, ρ_g, c_p are temperature diffusivity, heat conduction, density and heat capacity of the bubble gas, respectively, and δt is the characteristic time scale of the process. The heat conduction coefficient depends on the gas temperature (Vargaftik 1972) as

$$\frac{\lambda_g}{\lambda_{g0}} = \left(\frac{T}{T_0}\right)^{\alpha} \quad (\alpha \approx 0.8). \tag{8.7}$$

Taking into account (6.13) and that

$$\frac{T}{T_0} = \left(\frac{a}{a_0}\right)^{-3(\kappa-1)} \quad (1 < \kappa < \gamma), \quad \frac{\rho_g}{\rho_{g0}} = \left(\frac{a}{a_0}\right)^{-3}, \tag{8.8}$$

and neglecting by variation of heat capacity ($c_p \sim$ const) one can obtain the following estimation for the temperature diffusivity coefficient:

$$\frac{\nu_g^{(T)}}{\nu_{g0}^{(T)}} = \left(\frac{a}{a_0}\right)^{3(1.8-0.8\kappa)}. \tag{8.9}$$

Let us estimate the relative boundary layer thickness $\delta r^{(T)}/a$ for the bubble with equilibrium size $a_0 \sim 10^{-5}$ m oscillating in water ($\nu_{g0}^{(T)} = 10^{-6}\,\mathrm{m^2\,s^{-1}}$) with period $t_\omega \sim 10^{-5}$ s. For the low-Mach-number stage ($\delta t \sim t_\omega \sim 10^{-5}$ s, $a \sim a_0 \sim 10^{-5}$ m, $\nu_g^{(T)} \sim 10^{-6}\,\mathrm{m^2\,s^{-1}}$) the relative boundary layer thickness is $\delta r^{(T)}/a \sim 1$ and therefore the best approximation is an isothermal one ($\kappa \approx 1$).

For the same bubble ($a_0 \sim 10^{-5}$ m) during an implosion, i.e. the large-Mach-number stage, $\delta t \sim 10^{-8}$ s, $a \sim 10^{-6}$ m, the relative boundary layer thickness is $\delta r^{(T)}/a < 0.05$ and therefore the adiabatic approximation ($\kappa \approx \gamma$) may be used.

9. Linear harmonic forced oscillations

A priori it is clear that the problem has two characteristic frequencies. The first is the circular-flask frequency, ω_R, determined by the time, $t_R = 2R/C$, for acoustic wave propagation over the distance $2R$ from the flask wall to its centre and back; the second is the circular-bubble free oscillation frequency (i.e. the Minnaert frequency), ω_a:

$$\omega_R = 2\pi f_R = \frac{2\pi}{t_R} = \frac{\pi C}{R}, \quad \omega_a = \frac{1}{a_0}\sqrt{\frac{3\kappa_\sigma p_0}{\rho}}, \quad \kappa_\sigma = \kappa + \frac{3\kappa - 1}{3}\frac{2\sigma}{a_0 p_0}. \tag{9.1}$$

Let us consider the small oscillations of a bubble forced by flask oscillations. Introducing small disturbances:

$$\Delta a = a - a_0 \ll a_0, \quad \Delta p_I = p_I - p_0 \ll p_0, \quad \Delta p_R = p_R - p_0 \ll p_0, \tag{9.2}$$

and after linearization the equations for these disturbances may be written

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Delta a + \omega_a(\eta_\mu + \eta_C)\frac{\mathrm{d}}{\mathrm{d}t}\Delta a + \omega_a^2 \Delta a = \frac{\Delta p_I}{\rho a_0} - \frac{1}{\rho C}\frac{\mathrm{d}}{\mathrm{d}t}\Delta p_I, \tag{9.3}$$

$$\Delta p_I \,\big|_{t+R/C} = \Delta p_I \big|_{\,t-R/C} + \frac{2R}{C}\frac{\mathrm{d}}{\mathrm{d}t}\Delta p_F\bigg|_t + \frac{3\kappa_\sigma p_0}{C}\frac{\mathrm{d}}{\mathrm{d}t}\Delta a\bigg|_{t-R/C} \tag{9.4}$$

$$\left(\eta_\mu \equiv \frac{4\mu}{\rho a_0}\sqrt{\frac{\rho}{3\kappa_\sigma p_0}}, \quad \eta_C \equiv \frac{1}{C}\sqrt{\frac{3\kappa_\sigma p_0}{\rho}} \ll 1\right).$$

Here it was taken into account, that $\eta_\mu \eta_C \ll 1$.

Let us consider forced steady harmonic oscillations with a given arbitrary frequency ω:

$$\Delta p_R/p_0 = A_{pR}\mathrm{e}^{\mathrm{i}\omega t}, \quad \Delta p_I/p_0 = A_{pI}\mathrm{e}^{\mathrm{i}\omega t}, \quad \Delta a/a_0 = A_a \mathrm{e}^{\mathrm{i}\omega t}, \tag{9.5}$$

where A_{pR}, A_{pI}, A_a are small relative (non-dimensional) amplitudes of the flask wall pressure, incident pressure and bubble radius, respectively. We note that the amplitude of the flask pressure should be given.

Substituting the assumed harmonic solutions (9.5) into (9.3), (9.4) we obtain the equations for the relative amplitudes:

$$A_a = -\frac{K_a}{3\kappa_\sigma}A_{pI}, \quad K_a = \frac{\omega_a^2[1 + \mathrm{i}(\omega a_0/C)]}{\omega_a^2 + \mathrm{i}\omega_a(\eta_\mu + \eta_C)\omega - \omega^2}, \tag{9.6}$$

$$A_{pI} \sin\left(\frac{\omega R}{C}\right) = \left(\frac{\omega R}{C}\right) A_{pR} + 3\kappa_\sigma \left(\frac{\omega a_0}{C}\right) \exp\left(-\frac{\mathrm{i}\omega R}{C}\right) A_a. \tag{9.7}$$

Then the response function, which is the ratio of the relative amplitude of the bubble radius, A_a, to the the relative amplitude of the forcing flask pressure, A_{pR}, may be presented as a function of the non-dimensional flask frequency, $\overline{\omega}$, related to the flask resonant frequency, ω_R:

$$\frac{A_a}{A_{pR}} = -\frac{1}{3\gamma_\sigma}\frac{\pi\overline{\omega}[1 + \delta(\overline{\omega})]}{\sin \pi\overline{\omega} + \varepsilon_a \pi[1 + \delta(\overline{\omega})]\overline{\omega} \exp(-\pi\overline{\omega})}, \tag{9.8}$$

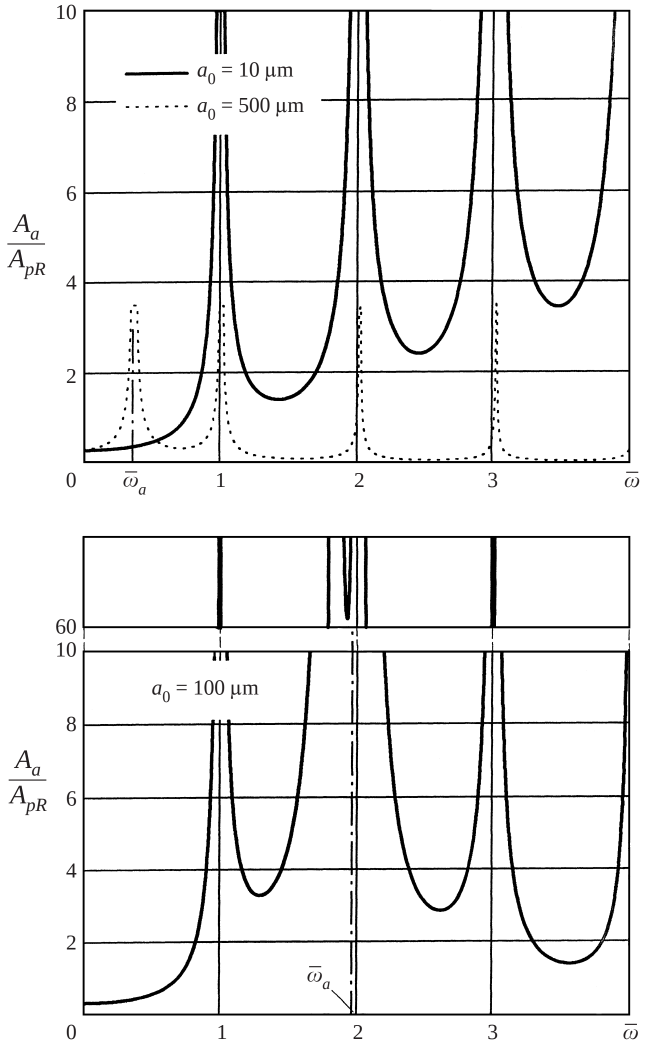

FIGURE 4. Amplitude–frequency response functions for harmonic oscillations of the isothermal ($\kappa = 1$) bubble ($a_0 = 10\,\mu$m ($\omega_a = 2160$ kHz), $a_0 = 100\,\mu$m ($\omega_a = 206$ kHz), $a_0 = 500\,\mu$m ($\omega_a = 41.03$ kHz)) in a flask ($R = 5$ cm, $\omega_R = \omega_1 = 94.2$ kHz) filled with water ($p_0 = 1$ bar).

$$\delta(\overline{\omega}) \equiv \frac{\varepsilon_\omega^2\overline{\omega}^2 + \mathrm{i}[\pi\varepsilon_a - (\eta_\mu + \eta_C)\varepsilon_\omega]\overline{\omega}}{1 - \varepsilon_\omega^2\overline{\omega}^2 + \mathrm{i}\varepsilon_\omega - (\eta_\mu + \eta_C)\overline{\omega}},$$

$$\overline{\omega} \equiv \frac{\omega}{\omega_R}, \quad \varepsilon_a \equiv \frac{a_0}{R}, \quad \varepsilon_\omega \equiv \frac{\omega_R}{\omega_a} \equiv \frac{a_0}{R}\frac{\pi C}{\sqrt{3\kappa_\sigma p_0/\rho}}.$$

Typical amplitude–frequency response functions corresponding to (9.8) are shown on figure 4 for a flask of radius $R = 5$ cm, which is filled with water. The first flask resonance frequency ($\overline{\omega} = 1$) corresponds to $f_1 = 15$ kHz (i.e. $\omega_1 = 94.2$ kHz).

For an extremely small bubble, which is small not only compared with the flask ($\varepsilon_a \ll 1$), but even ε_ω is small, and bearing in mind that usually $\eta_\mu < 1$, $\eta_C \ll 1$, it is easy to see that the resonance corresponding to the maximum absolute value of $|A_a/A_{pR}|$ takes place when $\sin(\pi\overline{\omega}) \approx 0$, or when $\overline{\omega} \approx 1, 2, 3, \ldots$. That is, the resonance is associated with the flask acoustic resonance, and the bubble does not influence the value of the resonant frequencies. This means that the ratio of the time of wave propagation from the flask to the centre and back, $2R/C$, to the period of the flask oscillations, f^{-1}, is an integer number κ:

$$\frac{2R/C}{f_k^{-1}} = k = 1, 2, 3, \ldots \quad \text{or} \quad \omega = \omega_k \equiv 2\pi f_k = \frac{k\pi C}{R} \equiv k\omega_R. \tag{9.9}$$

The bubble influences the resonant frequency only when the frequency of the flask excitation is comparable with the resonant frequency of the bubble, ω_a (see figure 4 for $a_0 = 100$ and $500\,\mu\text{m}$).

It is interesting that away from the bubble resonance zone (Minnaert frequency) the smaller the bubble the higher the relative response $|A_a/A_{pR}|$ (i.e. compare the curves for $a_0 = 10$ and $500\,\mu\text{m}$ on figure 4). This effect may be one of the possible reasons why sonoluminescence has only been observed for very small bubbles (e.g. $a_0 = 4$ to $10\,\mu\text{m}$).

It is seen that for frequencies close to the resonant frequencies the amplitude of the relative bubble radius oscillations, A_a, may be large even for small relative amplitudes of the flask pressure oscillations, $A_{pR} \ll 1$, because of very large values of $|A_a/A_{pR}|$ (see figure 4). This means that for excitation frequencies close to the resonant frequencies it is necessary to use nonlinear models when analysing the non-harmonic response of the bubble radius, Δa, which may take place even for small sinusoidal oscillations of the pressure on the flask, $\Delta p_R \ll p_0$, such as occurs in experiments on bubble sonoluminescence.

To analyse the influence of the bubble oscillations on the sound field let us consider the other type of response function, which is the ratio of the relative amplitude of the incident pressure, A_{pI}, to the relative amplitude of the forcing flask pressure, A_{pR}:

$$\frac{A_{pI}}{A_{pR}} = \frac{\omega R/C}{\sin(\omega R/C) + (\omega a_0/C)\exp(-\mathrm{i}\omega R/C)K_a}. \tag{9.10}$$

It is easy to see that, when $a_0 \to 0$, the influence of the bubble oscillations on the dynamics of the fluid vanishes and the response function (9.10) leads to the well-known solution that corresponds to the standing spherical acoustic wave (see Landau & Lifshitz 1959):

$$\frac{A_{pI}}{A_{pR}} = \frac{\omega R/C}{\sin(\omega R/C)}. \tag{9.11}$$

Here A_{pI} is the relative pressure amplitude in the centre of the flask without the bubble. In the case of flask resonance (see (9.9)) the pressure amplitude is infinite. However, when $a_0 \neq 0$, the infinite liquid pressure oscillations are compensated by the bubble oscillations (the denominator in (9.10) is never zero).

10. Nonlinear analysis for the resonant frequencies

For the flask resonance case, when $2R/C = k/f_k$ or $\omega = \omega_k \equiv k\pi C/R\,(k = 1, 2, 3, \ldots)$, one may write (because of periodicity with period $t_R = 2R/C$)

$$p_I\left(t + \frac{R}{C}\right) = p_I\left(t - \frac{R}{C}\right). \tag{10.1}$$

Then from (8.3) it follows that

$$\frac{2R}{C}\frac{\mathrm{d}}{\mathrm{d}t}[p_R(t)] = \frac{2\rho}{C}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[Q\left(t-\frac{R}{C}\right)\right]. \tag{10.2}$$

After integrating over time one may write

$$\frac{R}{\rho}[p_R(t) - p_0 c^{(1)}] = \frac{\mathrm{d}}{\mathrm{d}t}\left[Q\left(t-\frac{R}{C}\right)\right], \tag{10.3}$$

where $c^{(1)}$ is an arbitrary non-dimensional constant.

For the periodic regime excited by the sinusoidal oscillations of the flask wall with amplitude $\Delta p_R^o \equiv A_{pR}p_0$, at resonant frequency $\omega_k = k\pi C/R$,

$$p_R(t) = p_0(1 + A_{pR}\sin(\omega_k t)) \tag{10.4}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q(t)] = \frac{p_0 R}{\rho}\left\{A_{pR}\sin\left[\omega_k\left(t+\frac{R}{C}\right)\right] + 1 - c^{(1)}\right\}. \tag{10.5}$$

Keeping in mind that $Q(t) \equiv a^2 a' = \frac{1}{3}(a^3)'$ one may write after a double integration of (10.5)

$$\frac{a^3}{3} = \frac{Rp_0A_{pR}}{\rho\omega_k^2}\left\{-\sin\left[\omega_k\left(t+\frac{R}{C}\right)\right] + 1 + c^{(3)}\right\} + \frac{Rp_0}{\rho}\left\{\frac{c^{(2)}}{\omega_k}t + (1-c^{(1)})\frac{t^2}{2}\right\}, \tag{10.6}$$

where we have three arbitrary non-dimensional constants, $c^{(1)}, c^{(2)}$ and $c^{(3)}$. For periodic solutions it is necessary that $1 - c^{(1)} = 0$ and $c^{(2)} = 0$, thus

$$a^3(t) = a_*^3\left\{1 + c^{(3)} - \sin\left[\omega_k\left(t+\frac{R}{C}\right)\right]\right\}, \quad a_*^3 = \frac{3R^3A_{pR}p_0}{\pi^2k^2\rho C^2}. \tag{10.7}$$

It is seen that because $a > 0$, and $a_{\min}^3 = a_*^3 c^{(3)}$, it is necessary that $c^{(3)} > 0$.

The value of the parameter $c^{(3)}$ should be calculated from the flask radius periodicity condition.

It is interesting that equation (10.3) for the resonant regime with sinusoidal pressure excitation on the flask may be rewritten as

$$aa'' + 2(a')^2 = \frac{R}{a}\frac{p_R(t+R/C) - p_0}{\rho}. \tag{10.8}$$

This equation is valid only for periodic flask pressure oscillations having resonant frequencies, ω_k, and it looks similar to the Rayleigh–Plesset equation. However, this equation has a coefficient 2 instead $\frac{3}{2}$ in the term with $(a')^2$ and the driving pressure has the opposite sign. The coefficient (R/a) may be considered as an amplification factor due to the convergence of the pressure waves from the flask to the surface of the bubble (i.e. the cumulation effect). It is interesting that the properties of the gas (γ) and the initial conditions (a_0 and p_0) are manifested only through the flask radius or flask velocity periodicity conditions which may be presented as boundary conditions for (10.8) (also see the discussion of (12.4)–(12.6)).

The solution (10.7) yields a non-trivial and important radius, a_*, for the maximum value of the bubble radius, $a_{\max}$. This radius is a product of ‘large’ flask radius, R, and a small parameter $(\Delta p_R^o/\rho C^2)^{1/3}$, determined by the pressure amplitude on the flask, Δp_R^o.

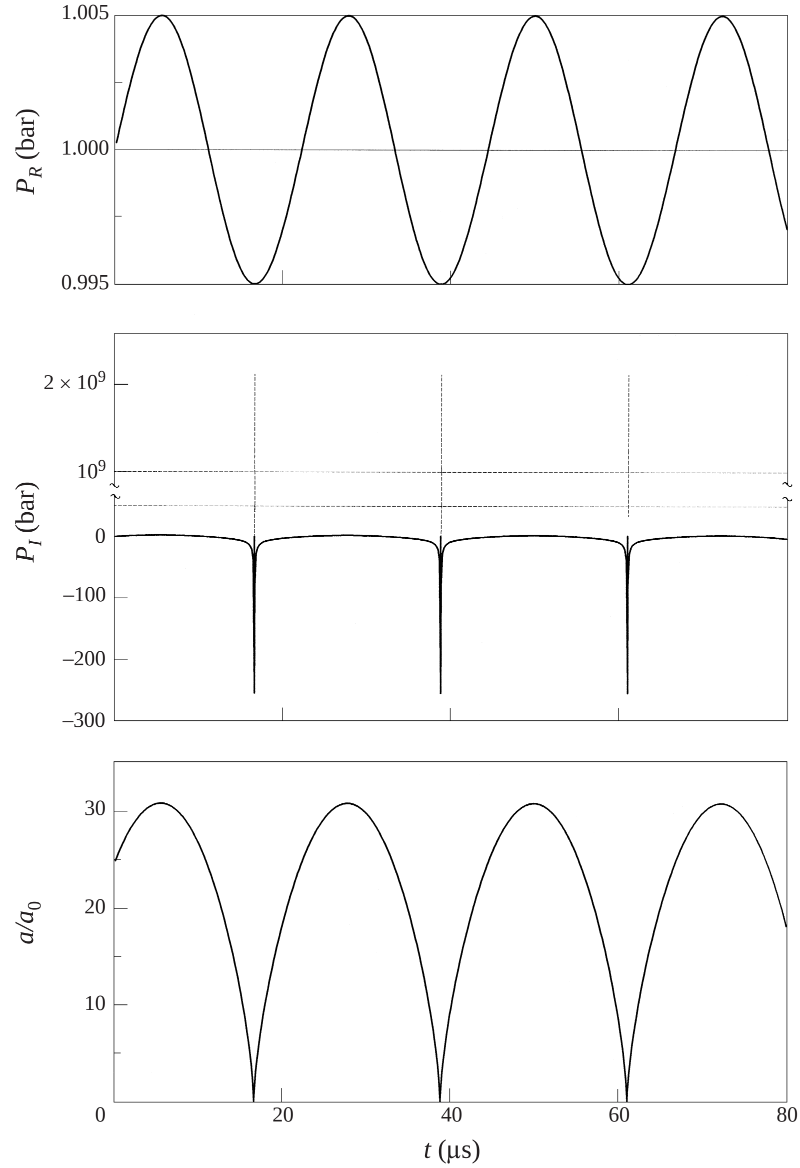

FIGURE 5. The periodic resonance regime of the isothermal ($\kappa = 1$) bubble oscillations ($a_0 = 4\,\mu$m) in water ($p_0 = 1$ bar, $T_0 = 300\,$K) in a spherical flask ($R = 5\,$cm), when the flask wall produces pressure sinusoidal oscillations of amplitude, $\Delta p_R^o = 0.005$ bar, and frequency, $f = f_3 = 45\,$kHz.

To have bubble implosion and sonoluminescence in the resonance regime it is necessary that

$$a_{\min} \ll a_0 \ll a_{\max} \sim a_* \equiv R \left(\frac{3\Delta p_R^o}{\pi^2 k^2 \rho C^2} \right)^{1/3}, \quad c^{(3)} \ll 1. \tag{10.9}$$

This means that, for the resonant regime, during an implosion the maximum value of the bubble radius, $a_{\max}$, does not depend on the initial value of the bubble radius, a_0. Then the maximum radius of the bubble and the compression ratio, $\eta_{\text{com}} \equiv a_{\max}^3/a_{\min}^3$, may be estimated by the formulae $a_{\max}^3 \approx 2a_*^3$, $\eta_{\text{com}} = 2/c^{(3)}$.

However, the values of the minimum radius of the bubble, $a_{\min}$, and the compression ratio, η_{com}, calculated here cannot be considered as physically realistic values during bubble collapse because the implosion period does not satisfy the long-wave and small-Mach-number approximation. Indeed, the implosion period needs a separate approach using a suitable numerical code (see § 13).

The periodic resonant regime is shown in figure 5. The calculations are given for the isothermal bubble. For the adiabatic bubble, the peaks of the incident pressure and the value of the maximum compression at oscillations of the bubble radius will be considerably less. For their other characteristics, the curves completely coincide.

Let us now consider the problem of periodic resonant oscillations when the displacement, δ_R, of the flask wall is given as a harmonic resonant oscillation with small amplitude δ_R^o and resonant frequency ω_k: $\delta_R = \delta_R^o \sin(w_k t), \delta_R^o \ll R$.

For the periodic regime the velocity of the flask wall, calculated by differentiating equation (5.6) for the velocity potential, φ_{ex}, yields

$$\delta_R^o \omega_k \cos(\omega_k t) = \frac{1}{R^2} Q\left(t - \frac{R}{C}\right) + \frac{1}{CR}\left[2\psi_2'\left(t - \frac{R}{C}\right) + Q'\left(t - \frac{R}{C}\right)\right]. \qquad (10.10)$$

After differentiating this equation with respect to time this equation may be rewritten as

$$\frac{1}{C}[2\psi_2''(t) + Q''(t)] = -\frac{1}{R}Q'(t) - R\delta_R^o\omega_k^2 \sin\left[\omega_k\left(t + \frac{R}{C}\right)\right]. \qquad (10.11)$$

Substituting this expression into (5.8) one may write the differential equation for bubble-radius resonant oscillations excited by flask-wall sinusoidal displacements in the following form:

$$aa'' + \tfrac{3}{2}a'^2 = \frac{p_a - p_\infty}{\rho}, \quad p_\infty = p_0 + \rho R \delta_R^o \omega_k^2 \sin\left[\omega_k\left(t + \frac{R}{C}\right)\right]. \qquad (10.12)$$

Here the smallness of the parameter (a/R) was used. That is why the components entering $Q'(t)/R$ are negligibly small compared with the corresponding components on the left-hand side of (10.11).

It is interesting that for acoustically resonant harmonic displacement oscillations of the flask wall periodic radial oscillations of the bubble are described by the Rayleigh–Plesset equation (for an incompressible liquid) with the pressure 'far from the bubble', p_∞, oscillating harmonically with amplitude $\Delta p_\infty^o = \rho R \delta_R^o \omega_k^2 \sim \pi k \Delta p_R^o$.

Here in estimating Δp_∞^o it was taken into account that $w_R^o = \delta_R^o \omega_k$ is the amplitude of the velocity on the flask wall and $\rho\omega_R^o C \sim \Delta p_R^o$. Then coefficient πk determines the efficiency or amplification factor of the pressure amplitude on the flask Δp_R^o for the resonance of the kth order. To have bubble implosion and sonoluminescence it is necessary (but not sufficient) to have $\Delta p_\infty^o \gtrsim p_0$, that is

$$\pi k \Delta p_R^o \gtrsim p_0. \qquad (10.13)$$

11. Kinetic energy evolution for the resonance regime

It is interesting to calculate the kinetic energy of the liquid concentrated near the bubble, where the velocity distribution is given by (4.2):

$$k = \frac{1}{2}\int_a^{r_o} 4\pi r^2 \rho w^2 \, \mathrm{d}r \approx 2\pi\rho \int_a^{\infty} \left(\frac{w_a a^2}{r^2}\right)^2 r^2 \, \mathrm{d}r = 2\pi\rho a^3 (w_a)^2. \qquad (11.1)$$

For the resonant periodic regime equation (10.7) yields

$$w_a = a' = a_* \frac{\mathrm{d}}{\mathrm{d}t} \left\{ 1 + c^{(3)} - \sin\left[\omega_k \left(t + \frac{R}{C}\right)\right]\right\}^{1/3}$$

$$= -\frac{a_* \omega_k}{3} \left\{ 1 + c^{(3)} - \sin\left[\omega_k \left(t + \frac{R}{C}\right)\right]\right\}^{-2/3} \cos\left[\omega_k \left(t + \frac{R}{C}\right)\right]. \quad (11.2)$$

Thus the kinetic energy of the liquid, K, may be expressed as

$$K = \frac{2\pi\rho a_*^5 \omega_k^2}{9} \frac{\cos^2 \Omega}{\{1 + c^{(3)} - \sin\Omega\}^{1/3}} \quad \left(\Omega \equiv \omega_k \left(t + \frac{R}{C}\right)\right). \quad (11.3)$$

To calculate the maximum value of the kinetic energy it is necessary to calculate the derivative with respect to time:

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{2\pi\rho a_*^5 \omega_k^2}{9} \frac{\cos\Omega[-2\sin\Omega\{1 + c^{(3)} - \sin\Omega\} + \frac{1}{3}\cos^2\Omega}{\{1 + c^{(3)} - \sin\Omega\}^{4/3}}. \quad (11.4)$$

Equating this expression to zero, one obtains two equations for optimal values of K:

$$\cos\Omega = 0, \quad \sin^2\Omega - \tfrac{6}{5}(1 + c^{(3)})\sin\Omega + \tfrac{1}{5} = 0. \quad (11.5)$$

The first equation and its roots are a trivial solution which corresponds to the minimum value of the liquid kinetic energy, $K = 0$. The second equation and its real root corresponds to the maximum value of K: $(\sin\Omega)_1 = 1 + c^{(3)} > 1 (\sin\Omega)_2 = \frac{1}{5}(1 + c^{(3)})$.

Thus the maximum value of the liquid kinetic energy is

$$K_{\max} = \frac{2\pi\rho a_*^5 \omega_k^2}{9} \frac{\frac{24}{25}(1 - \frac{1}{12}c^{(3)} - \frac{1}{24}(c^{(3)})^2)}{\{\frac{4}{5}(1 + c^{(3)})\}^{1/3}}. \quad (11.6)$$

It is seen that for bubble implosion, when $c^{(3)} \ll 1$, the maximum value of the kinetic energy does not depend on the constant $c^{(3)}$ or on the properties of the gas and initial radius of the bubble a_0:

$$K_{\max} \approx \frac{2\pi\rho a_*^5 \omega_k^2}{9} \approx \frac{8.52}{k^{4/3}} \left(\frac{\Delta p_R^o}{\rho C^2}\right)^{5/3} \rho R^3 C^2. \quad (11.7)$$

It should be noted that $K_{\max}$ is the energy that can be used for compression of the gas and the liquid at the moment of the maximum compression of the gas bubble.

The characteristic values of the bubble radius a^* and the velocity on the interface w^* at the moment of the maximum value of the kinetic energy ($\sin\Omega \approx \frac{1}{5}$) are

$$a^* \approx 0.93 a_*, \quad w^* \approx -0.387 a_* \omega_k \approx -0.817 k^{1/3} \left(\frac{\Delta p_R^o}{\rho C^2}\right)^{1/3} C. \quad (11.8)$$

It is interesting to note that for the resonance regime under consideration, the interface velocity w_0 at the moment when the bubble radius coincides with the initial value a_0 equals

$$w_0 \approx -0.471 \omega_k \sqrt{\frac{a_*^3}{a_0}} = -0.260 \left(\frac{R_0}{a_0}\right)^{1/2} \left(\frac{\Delta p_R^o}{\rho C^2}\right)^{1/2} C. \quad (11.9)$$

12. Nonlinear analysis for the near-resonant frequencies – small deviation from resonance

Let us consider periodic oscillations with frequencies close to the resonant frequency:

$$\omega = \omega_k + \Delta\omega, \quad \Delta\omega \ll \omega_k \quad (\omega_k = \pi k C/R, \quad k = 1, 2, 3 \cdots). \tag{12.1}$$

The periodicity condition implies

$$\begin{aligned}\psi_2'\left(t+\frac{R}{C}\right) &= \psi_2'\left(t+\frac{R}{C}-k\frac{2\pi}{\omega}\right) = \psi_2'\left(t+\frac{R}{C}-k\frac{2\pi}{\omega_k+\Delta\omega}\right)\\ &= \psi_2'\left(t+\frac{R}{C}-\frac{2\pi k}{\omega_k}\left(1-\frac{\Delta\omega}{\omega_k}\right)\right)+O\left(\left(\frac{\Delta\omega}{\omega_k}\right)^2\right)\\ &= \psi_2'\left(t-\frac{R}{C}+\frac{2\pi k}{\omega_k}\frac{\Delta\omega}{\omega_k}\right)\\ &= \psi_2'\left(t-\frac{R}{C}\right)+\frac{2\pi k}{\omega_k}\frac{\Delta\omega}{\omega_k}\psi_2''\left(t-\frac{R}{C}\right)+O\left(\left(\frac{\Delta\omega}{\omega_k}\right)^2\right).\end{aligned} \tag{12.2}$$

Next, equation (6.9) may be rewritten as

$$\left.\begin{aligned} p_R(t) = {} & p_0 - \frac{\rho}{R}\left[\frac{2R}{C}\frac{\Delta\omega}{\omega_k}\psi_2''\left(t-\frac{R}{C}\right) - Q'\left(t-\frac{R}{C}\right)\right] \\ \text{or} & \\ \frac{2}{C}\psi_2''(t) = {} & \frac{\omega_k}{\Delta\omega}\frac{a}{R}(aa''+2a'^2) - \frac{\omega_k}{\Delta\omega}\frac{p_R(t+R/C)-p_0}{\rho}. \end{aligned}\right\} \tag{12.3}$$

In this case the result depends on a non-dimensional parameter characterizing the influence of the bubble on the near-resonant oscillations:

$$A_* = \frac{a_0}{R}\frac{\omega_k}{\Delta\omega}.$$

This parameter is a product of small (a_0/R) and large ($\omega_k/\Delta\omega$) parameters.

Taking into account (12.3), equation (5.8) may be rewritten in the following form:

$$aa'' + \frac{3}{2}a'^2 = \frac{p_a - p_0}{\rho} - \frac{\omega_k}{\Delta\omega}\frac{p_R(t+R/C)-p_0}{\rho} + \frac{\omega_k}{\Delta\omega}\frac{a}{R}(aa''+2a'^2) + \frac{1}{C}Q''(t). \tag{12.4}$$

The pressure inside the bubble may be greater than the driving pressure $(p_a - p_0) \sim \omega_k \Delta p_R/\Delta\omega$, thus the first two terms on the right-hand side are dominant terms. If

$$\frac{a}{R}\frac{\omega_k}{\Delta\omega} \ll 1, \tag{12.5}$$

then the last two terms in (12.4) are small and one can use the approximation that was used in § 6:

$$Q' \cong a\left[\frac{a'^2}{2} + \frac{p_a - p_0}{\rho} - \frac{\omega_k}{\Delta\omega}\frac{\Delta p_R(t+R/C)}{\rho}\right].$$

Differentiating this expression with respect to time one can derive from (12.4) the following ordinary differential equation for near-resonant periodic forced oscillations of the bubble in the flask:

$$\left(1-\frac{a'}{C}-\frac{a}{R}\frac{\omega_k}{\Delta\omega}\right)a\frac{\mathrm{d}^2a}{\mathrm{d}t^2}+\left(\tfrac{3}{2}-\frac{a'}{2C}-2\frac{a}{R}\frac{\omega_k}{\Delta\omega}\right)\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2=\left(1+\frac{a'}{C}\right)$$
$$\times\left[\frac{p_a-p_0}{\rho}-\frac{1}{\rho}\frac{\omega_k}{\Delta\omega}\Delta p_R\left(t+\frac{R}{C}\right)\right]+\frac{a}{\rho C}\frac{\mathrm{d}}{\mathrm{d}t}\left[p_a-p_0-\frac{\omega_k}{\Delta\omega}\Delta p_R\left(t+\frac{R}{C}\right)\right]. \quad (12.6)$$

Typical solutions of equation (12.6) are shown on figure 6 for different flask pressure amplitudes. The figure shows that the oscillations of the bubble radius both for its adiabatic and isothermal behaviour differ quantitatively, but not qualitatively. The corresponding curves for the incident pressure and the pressure at local infinity are given in figure 7. The dependence of $a_{\min}$ and $a_{\max}$ on the frequency shift from the third flask resonance for $a_0 = 4\,\mu\mathrm{m}$ and different pressure amplitudes is shown on figure 8 for the isothermal bubble. The resonant case ($\Delta\omega = 0$) is denoted by the dark dots. In the vicinity of the flask resonance the curves are given by the dashed lines and equation (12.6) is not valid.

If should be noted that the near-resonant ($\Delta\omega \ll \omega_k$) regime, which is described by (12.6), corresponds to limitation (12.5):

$$\frac{a}{R} \ll \frac{\Delta\omega}{\omega_k} \ll 1, \quad (12.7)$$

and is quite different from the exact resonance ($\Delta\omega = 0$, $\omega = \omega_k$), described by analytical solution (10.7). To have the near-resonant regime close to the exact resonance it is necessary to have extremely small frequency shift:

$$\frac{\Delta\omega}{\omega_k} \ll \frac{a}{R} \ll 1. \quad (12.8)$$

For single-bubble sonoluminescence experiments with a small bubble ($a/R \sim 10^{-4}$) the tuning of frequency and other parameters of the system to have exact, or almost exact, resonant bubble excitation is hardly attainable except when gravity is absent, since for this case bubble displacement from the centre of the flask is excluded.

13. The peak shock waves emitted by the collapsing bubble

The theory presented in this paper is valid only for relatively slow evolution of the bubble. During implosion, which comprises only a very small part of the total period, the approximation of liquid incompressible near the bubble boundary layer is no longer a good approximation. Estimations and numerical calculations (Lofstedt, Barber & Putterman 1993; Moss *et al.* 1994) have shown that during an implosion the pressure in and near the bubble may be of order 10^6 bar, the density of the water may increase by 2 to 3 times, and the interfacial Mach number, M_a, may be larger than unity. This means that for bubble implosion it is necessary to take into account the nonlinear compressibility of the liquid near the bubble, shock wave formation and shock interaction both in the liquid and in the gas, convergence of the shock waves to the centre of the bubble and their cumulation (Wu & Roberts 1993; Moss *et al.* 1994). Significantly, shock wave cumulation may generate a superhigh pressure and temperature near the centre of the bubble. This process must be considered by a separate numerical (i.e. HYDRO) code based on the partial differential conservation equations and nonlinear equations of state with plasma and light radiation effects (Moss *et al.* 1994).

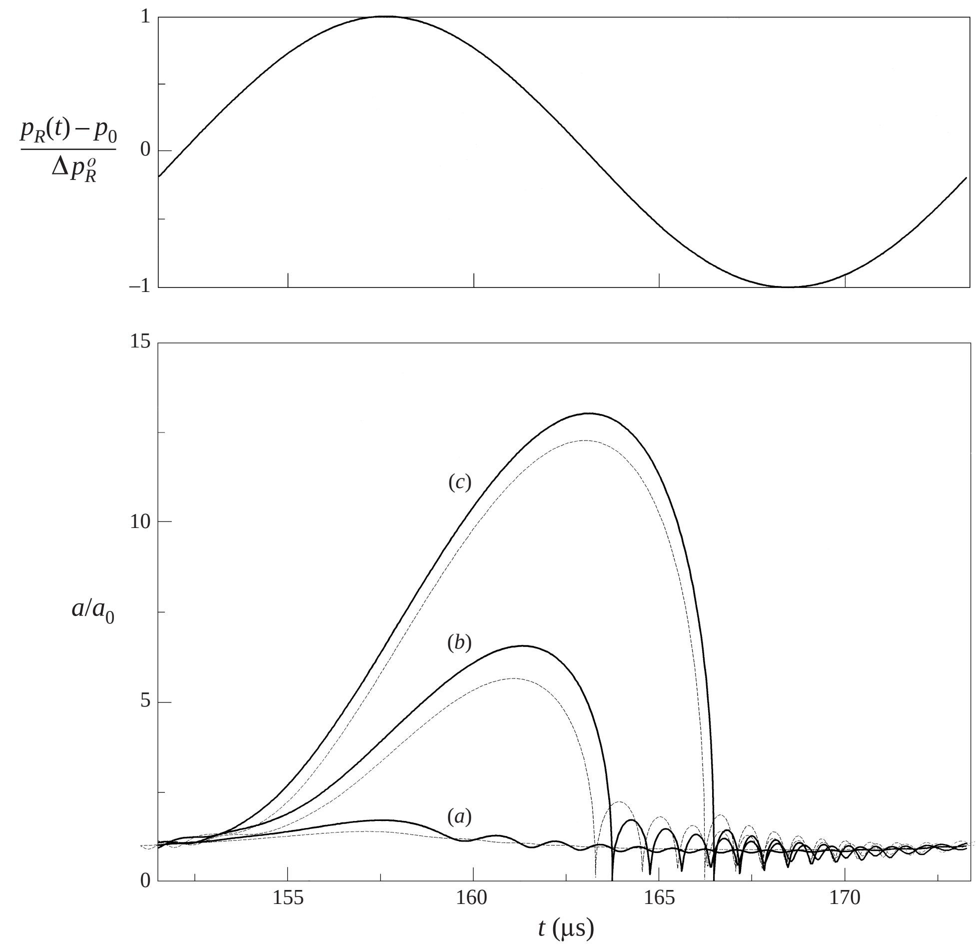

FIGURE 6. Excitation of gas bubble ($a_0 = 4\,\mu$m) oscillations in a flask ($R = 5$ cm) filled with water ($p_0 = 1$ bar, $T_0 = 300$ K) when the flask wall produces pressure sinusoidal oscillations with the frequency, $f = 46$ kHz ($f \approx f_3$, $\Delta f = 1$ kHz, $\Delta\omega = 6.28$ kHz) and different pressure amplitudes: (a) $\Delta p_R^o = 0.02$ bar, (b) $\Delta p_R^o = 0.03$ bar, (c) $\Delta p_R^o = 0.04$ bar. Solid and dashed lines correspond isothermal ($\kappa = 1$) and adiabatic ($\kappa = \gamma$) gas, respectively.

Fortunately the numerical code for the implosion period need consider only a very thin, strongly compressible, liquid boundary layer of thickness $\delta r_s = C\delta t_s \sim$ 10–$10^2\,\mu$m. On the external surface of this layer, $r = a + \delta r_s$, the solution presented in this paper may be used as a boundary condition. Such a strategy can reduce the time required for numerical calculations by a few orders of magnitude.

The almost ‘catastrophic’ collapse of the bubble during an implosion, induces a superhigh pressure in the gas bubble, and the rapidly expanding rebound of the bubble emits extremely short-duration ($\delta t_s < 10^{-8}$ s) pressure waves propagating from the centre of the flask to the wall of the flask during the low-Mach-number stage. These waves must be considered separately, taking into account the nonlinear compressibility of the liquid, since the sound speed increases with pressure, which greatly intensifies attenuation of the shock waves. In contrast, for linear acoustic theory (i.e. linear compressibility of the liquid) we know that the shock waves attenuate only due to the spherical divergence.

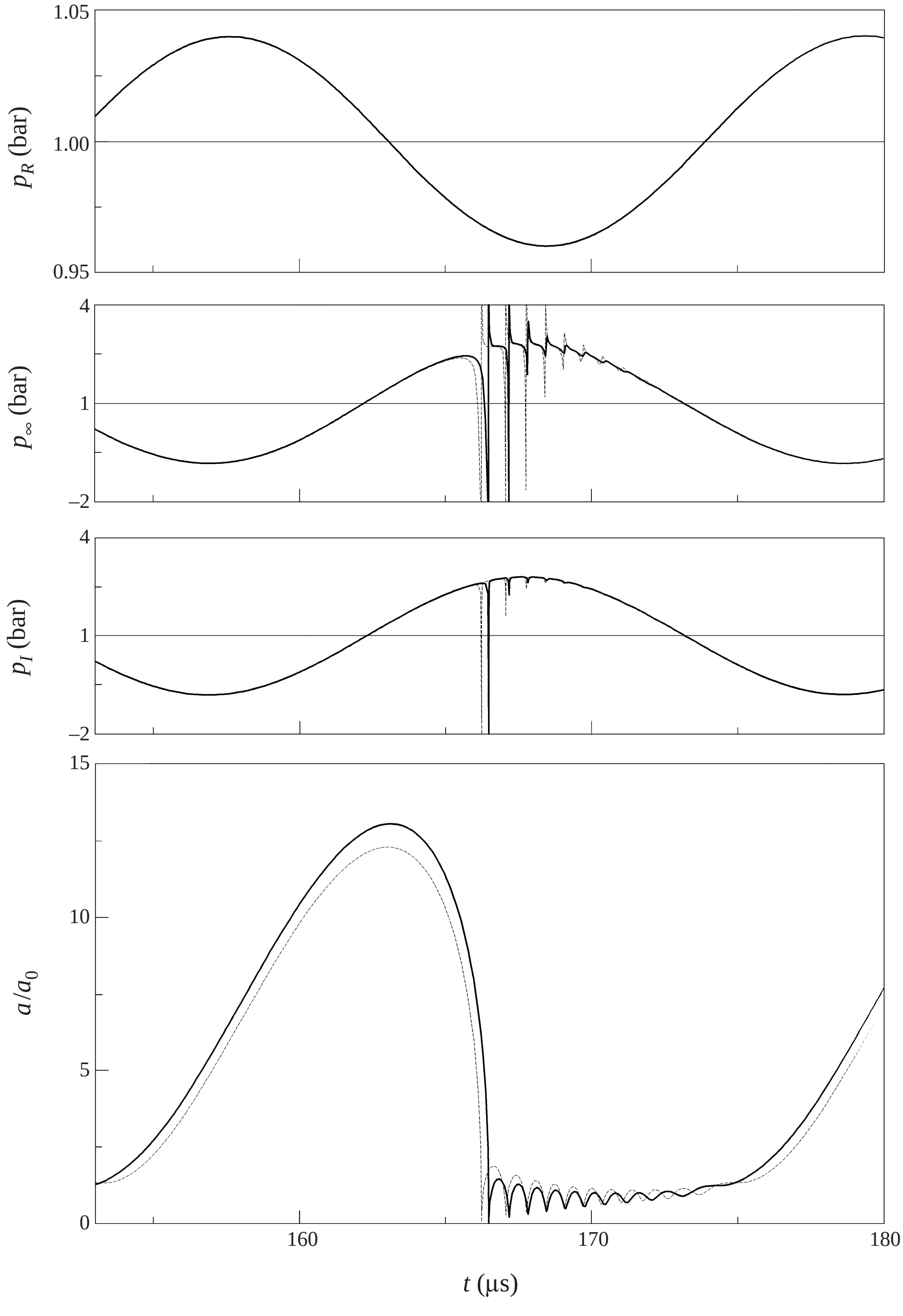

FIGURE 7. The same as figure 6 (line *c*) but showing the evolution of the incident pressure and the pressure at local infinity.

The contribution of nonlinear compressibility to wave attenuation can be understood by considering the nonlinear Boussinesq equation for a simple plane wave propagating in the x-direction without reflections, with a velocity which is close to the sound speed, C. The attenuation of the peak of the pressure, $p_{\max}$, for a wave

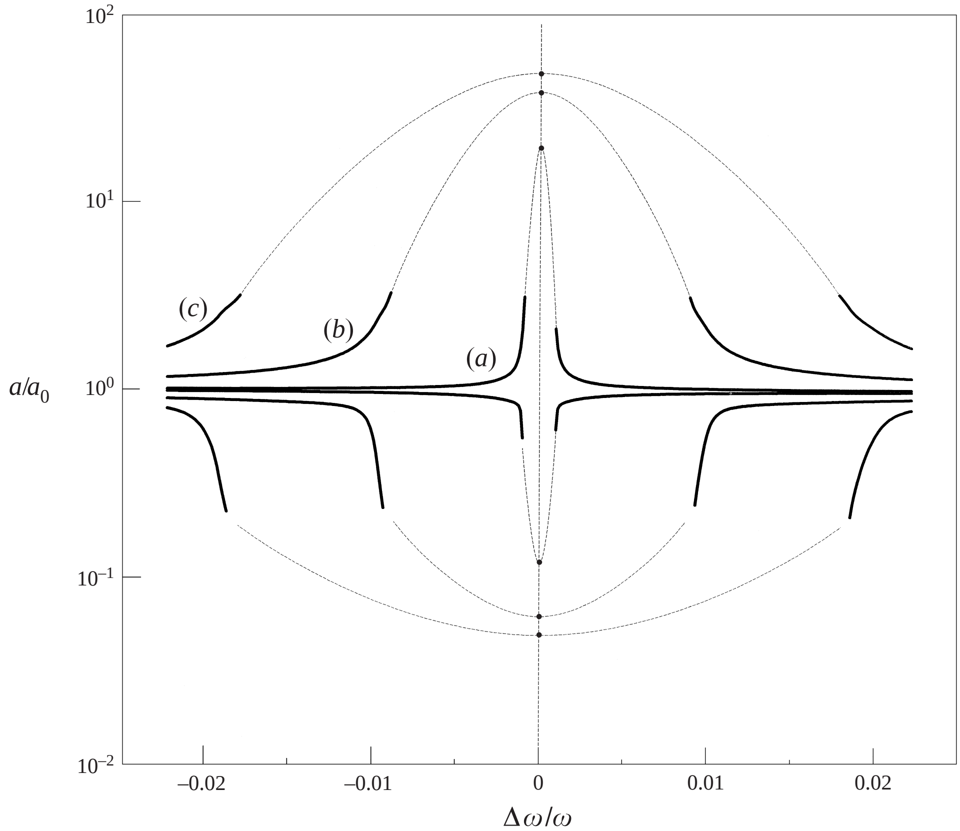

FIGURE 8. The dependence of a_{min} and a_{max} on frequency shift from the third flask resonance ($\omega = \omega_3 + \Delta\omega$) for $a_0 = 4\,\mu$m and different pressure amplitudes. (*a*) $\Delta p_R^o = 0.001$ bar, (*b*) $\Delta p_R^o = 0.01$ bar, (*c*) $\Delta p_R^o = 0.02$ bar.

having the initial duration, τ^o, is given by (Whitham 1974)

$$\frac{p_{\max}}{p_{\max}^o} = \frac{1}{\sqrt{1 + (x/\lambda^o)}}, \quad \lambda^o = \frac{\rho C^3}{\beta} \frac{\tau^o}{p_{\max}^o}, \quad \beta \equiv \frac{d^2 p}{dv^2} > 2\rho^3 C^2, \quad v \equiv \frac{1}{\rho}. \tag{13.1}$$

Along the direction of wave propagation, products of the amplitude of the wave and its duration τ or length λ are constant:

$$p_{\max}\tau = p_{\max}^o \tau^o, \quad p_{\max}\lambda = p_{\max}^o \lambda^o. \tag{13.2}$$

For water having $p_{\max}^o = 100$ bar, and $\tau^o = 10^{-8}$ s, the characteristic length of the attenuation is $\lambda^o = 3$ mm. This analysis shows that nonlinear compressibility of the liquid increases the attenuation of the peak waves initiated by the collapsing bubble, and these peak waves become quite weak before reaching the flask wall.

14. Summary

(i) There are two stages of the bubble's oscillation process. The first one is a low-Mach-number stage when the velocity of the bubble's interface is small compared with the sound speed in the liquid. The second stage, bubble implosion, is a stage of very rapid bubble collapse and gas compression, followed by a rebounding expansion.

During collapse and rebound the velocity of the interface may be comparable to, or larger than, the local liquid sound speed. Significantly, the low-Mach-number period takes up almost all the time of the overall process. Moreover, implosion takes place at, or near, the flask's acoustic resonant frequency, and takes place in a very short time ($< 10^{-8}$ s).

(ii) Two asymptotic solutions have been derived which are valid for the low-Mach-number regime. The first one is an asymptotic solution for the field far from the bubble, and it corresponds to a linear hyperbolic wave equation of second order. The second one is an asymptotic solution for the field near the bubble and corresponds to the Laplace equation for an incompressible fluid.

(iii) The low-Mach-number stage of the forced oscillations of a bubble in a compressible liquid may be described by the Rayleigh–Plesset equation, where the driving pressure is the pressure at local infinity p_∞ or the Herring equation, where the driving pressures is the incident pressure p_I. These driving pressures, p_∞, and p_I, are different from each other and from the pressure on the flask wall, $p_R(t)$. The driving pressures p_∞ and p_I may be calculated from the flask-wall pressure evolution $p_R(t)$ or from the flask-wall velocity evolution, $w_R(t)$, using ordinary difference-differential equation having lagging and leading time.

(iv) For the low-Mach-number stage for small gas bubbles ($a \lesssim 10^{-5}$ m) oscillations have frequencies of $\omega \lesssim 10^5\,\mathrm{s}^{-1}$, which are typical of single-bubble sonoluminescence experiments, isothermal gas bubble expansion and compression occurs ($\kappa \approx 1$).

In contrast, during the implosion stage ($\delta t \sim 10^{-8}$ s) there is transition to adiabatic compression and expansion of the gas bubble.

During the low-Mach-number stage of gas bubble oscillation in an acoustic field interfacial heat transfer weakly influences the bubble radius evolution, $a(t)$, but strongly influences the evolution of gas temperature evolution, $T(t)$.

(v) The analysis of an initial value problem for initiation of bubble oscillations by flask excitation reveals a very strong and curious evolution of the oscillations, and an attempt to obtain the periodic regime using the direct numerical codes based on partial differential conservation equations is not an effective procedure because one needs to calculate many evolving oscillations. Fortunately, the periodic process may be analysed analytically.

(vi) In the case of small harmonic oscillations, the response function, which is equal to the ratio of the relative amplitude of the bubble radius a to the relative amplitude of the flask's forcing pressure p_R, depends on the flask frequency ω. The maxima of this function determines the resonance of the bubble oscillations and they are associated with the acoustic flask resonance ($\omega = \omega_k$), when during the time of pressure wave propagation from the flask to the centre and back an integer number (k) of oscillations takes place (i.e. $2R/C = 2\pi k/\omega_k$).

It is interesting that away from the bubble's Minnaert frequency resonance zone the smaller the bubble the higher the relative response of the bubble to flask excitation. This effect may be one of the possible reasons why sonoluminescence has only been observed for very small bubbles ($a_0 = 4$ to $10\,\mu$m).

(vii) For near-resonant frequencies it is necessary to use nonlinear solutions. For such frequencies the bubble oscillations are not harmonic and implosions may occur even when oscillations of the flask wall have extremely small amplitudes. This is explained by the amplification of the convergent acoustic waves initiated by flask motion.

(viii) Two near-resonant asymptotic solutions ($\Delta\omega \equiv \omega - \omega_k \ll \omega_k$) were found.

The first one (near the flask's exact acoustic resonance) corresponds to an extremely small frequency shift from the flask's acoustic resonant frequency ($\Delta\omega/\omega_k \ll a/R$). For most practical situations this regime is not attainable. Indeed, if one wished to achieve this resonant regime it would only be possible in a situation in which gravity is absent, and the bubble does not experience displacement from the centre of the flask.

The second near-resonant asymptotic solution corresponds to a situation which is practically attainable: $\Delta\omega/\omega_k \gg a/R$.

(ix) The maximum radius of the bubble for the exact resonant periodic regime with implosion does not depend on the initial radius of the bubble a_0 and is determined by (10.7). Interestingly, for a small deviation of the frequency ω from the resonant one ω_k, the influence of a_0 may be noticeable.

(x) The amplitude of the pressure at the 'local infinity' of the bubble Δp_∞^o and the incident pressure Δp_I^o may be much larger than the amplitude of the pressure on the flask, Δp_R^o. This is due to amplification of the acoustic waves from the flask wall due to spherical convergence, and this amplification is fully or partly compensated by the expansion and compression of the bubble. This compensation is especially strong at resonant frequencies.

(xi) A threshold character is exhibited by either a frequency shift, $\Delta\omega \equiv \omega - \omega_k$, or a change in pressure amplitude on the flask's wall, Δp_R^o, when near resonant excitation.

(xii) For calculation of bubble implosion it is necessary to take into account the nonlinear compressibility of the liquid, and shock wave formation, but only in a small sphere within a radius of about ten radii of the bubble (i.e. $10a$).

(xiii) The 'catastrophic' collapse of the bubble during an implosion causes a super-high pressure in the gas bubble. The expanding rebound of the bubble and extremely short time duration associated with this expansion ($\delta t_s < 10^{-8}$ s), causes high-pressure shocks wave to be emitted, which propagate from the centre of the flask to the wall of the flask. According to linear acoustic theory this strong shock wave would attenuate only due to spherical divergence and would have a high enough amplitude to strongly shock the flask wall. However, due to nonlinear compressibility of the liquid, where sound speed increases with the pressure, the attenuation of the shock wave is very strong, and the shock wave becomes very weak before it reaches the flask wall, producing only some additional high-frequency 'noise'.

(xiv) More study of the picosecond processes of shock wave cumulation in the centre of the bubble and the resultant high gas temperatures is needed. This will require the use of suitable HYDRO codes and nonlinear equations of state for gas and liquid.

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